Exercise 88

A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2t^2}$, $t \geq 0$, where b and c are positive constants.

(a) Find the velocity and acceleration functions.

(b) Show that the particle always moves in the positive direction.

Solution

Part (a)

The velocity is the derivative of the position function.

$$\begin{aligned} v(t) &= \frac{dx}{dt} \\ &= \frac{d}{dt}\sqrt{b^2 + c^2t^2} \\ &= \frac{1}{2}(b^2 + c^2t^2)^{-1/2} \cdot \frac{d}{dt}(b^2 + c^2t^2) \\ &= \frac{1}{2}(b^2 + c^2t^2)^{-1/2} \cdot (2c^2t) \\ &= \frac{c^2t}{\sqrt{b^2 + c^2t^2}} \end{aligned}$$

The acceleration is the derivative of the velocity function.

$$\begin{aligned} a(t) &= \frac{dv}{dt} \\ &= \frac{d}{dt}\left(\frac{c^2t}{\sqrt{b^2 + c^2t^2}}\right) \\ &= \frac{\left[\frac{d}{dt}(c^2t)\right]\sqrt{b^2 + c^2t^2} - \left[\frac{d}{dt}\left(\sqrt{b^2 + c^2t^2}\right)\right]c^2t}{b^2 + c^2t^2} \\ &= \frac{(c^2)\sqrt{b^2 + c^2t^2} - \left[\frac{1}{2}(b^2 + c^2t^2)^{-1/2} \cdot \frac{d}{dt}(b^2 + c^2t^2)\right]c^2t}{b^2 + c^2t^2} \\ &= \frac{c^2\sqrt{b^2 + c^2t^2} - \left[\frac{1}{2}(b^2 + c^2t^2)^{-1/2} \cdot (2c^2t)\right]c^2t}{b^2 + c^2t^2} \end{aligned}$$

Simplify the right side.

$$a(t) = \frac{c^2\sqrt{b^2 + c^2t^2} - \frac{c^4t^2}{\sqrt{b^2 + c^2t^2}}}{b^2 + c^2t^2}$$

$$= \frac{\frac{c^2(b^2 + c^2t^2) - c^4t^2}{\sqrt{b^2 + c^2t^2}}}{b^2 + c^2t^2}$$

$$= \frac{\frac{c^2b^2}{\sqrt{b^2 + c^2t^2}}}{b^2 + c^2t^2}$$

$$= \frac{c^2b^2}{(b^2 + c^2t^2)^{3/2}}$$

Part (b)

The particle is always moving in the positive direction because the velocity is never negative: $t \geq 0$, and the square root yields a positive number.

$$v(t) = \frac{c^2t}{\sqrt{b^2 + c^2t^2}} \geq 0$$